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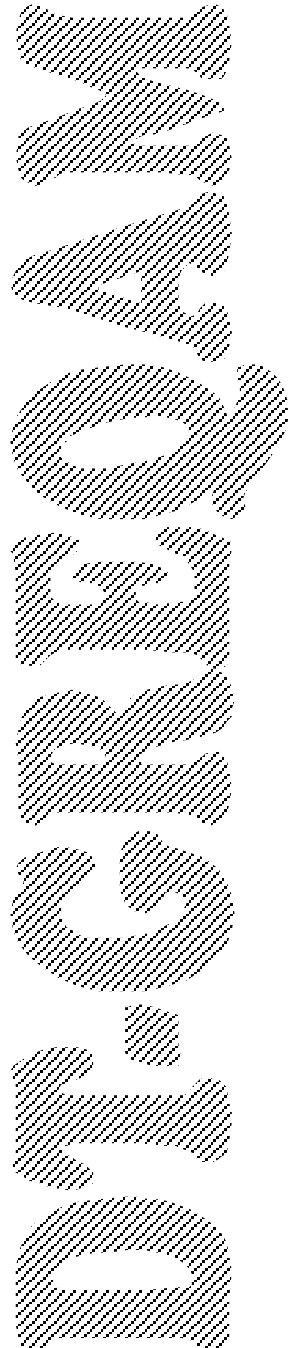
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## **A DROP OF RAINWATER AGAINST A DROP OF GROUNDWATER : Does Rainwater Harvesting really allow us to spare Groundwater?**

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# A drop of Rainwater against a drop of Groundwater : Does Rainwater Harvesting allows us really to spare groundwater?

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## Abstract

This paper is concerned with groundwater management issues in the presence of rainwater harvesting (RWH). Namely, we propose a two-state model in order to take into account the standard dynamics of the aquifer and the dynamics of the storage capacity since the collected rainwater reduces the natural recharge. We analyze the trade-off between these two water harvesting techniques in an optimal control model. We notably show that, when these techniques are pure substitutes, the development of RWH conducts in the long run to a depletion of the water table even if pumping is reduced.

Keywords: *Rainwater Harvesting, Conjunctive Use, Groundwater Optimal Control Management, Dynamic Model*

JEL: Q25, C61, D61

## Introduction

The issue of water management remains a major resource challenge of the 21<sup>st</sup> century registered in the Millennium Development Goals [39]. Such a context motivated this paper which examines to what extend groundwater resources can be jointly exploited with rainwater harvesting without unduly depleting the aquifer.

Rainwater Harvesting (abbreviated as RWH) is usually employed as an umbrella term describing a range of methods of collecting and conserving various forms of

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run off water. Quoted by Myers [24], Geddes was probably the first to define, in 1963, RWH as “the collection and storage of any form of water either runoff or creek flow for irrigation use” and Myers said that it represents “the practice of collecting water from area treated to increase runoff from rainfall or snowmelt”. Currier refers it as “the process of collecting natural precipitation from prepared watersheds for beneficial use”. In its broadest sense, RWH can be defined as the collection of water for its productive use [37]. Despite the few current literature dealing with this topic, the collection of rainwater used to be adopted by many civilizations and, therefore, is considered as the oldest technology developed by man to provide water. On the basis of the review of Gauthier [9], we can notice that the first evidence dates back as 6,000 years in the Gangsu region of China. Namely, rain tank storage was particularly important in Southern India where dams were built by the villagers to capture rainwater from the monsoons [32]. Yet, even if globally rainwater harvesting has decreased since the 1980s, and traditional systems have been neglected in favor of large-scale projects, as late as the early 20th century, rainwater was predominantly used in small islands [19] with no significant river systems and in remote and arid locations [28]. For instance, Gibraltar has one of the largest rainwater collection systems in existence and rainwater is still the primary water source on many US ranches. With the expected water crisis, RWH is even a technique enjoying a revival in popularity. For instance, in the Gangsu province - China, the Gangsu Research launched projects for water conservancy and by 2000, a total of 2,183,000 rainwater tanks had been built with a total of 73.1 million cubic meters supplying drinking water and supplementary irrigation. Perrens [29] estimates that in Australia approximately one million people rely on rainwater as their primary source of supply. RWH for potable use also occurs in rural areas of Canada and Bermuda [7].

Given these observational evidences, this paper aims at studying the conjunctive use of groundwater and RWH within an optimal control framework. Namely, we propose a two-state model with pure state constraints in order to take into account the standard dynamics of groundwater and the dynamics of RWH which is assimilated to a capital accumulation law. Actually, rainwater can be collected into various reservoirs such as rain tanks but all this equipment require, in any case, the development of a harvesting capacity through progressive investment.

This RWH capacity does however not work like a backstop technology in the sense of Heal [13], Dasgupta and Heal [6]. Recent studies extend this approach to groundwater resource (see for instance Kim and *al.* [17], Krulce and *al* [20], Holland and Moore [21], or Koundouri and Christou [16]). In fact, most of these papers, in the best of our knowledge, assume the existence of an alternative water source, often available at a constant average cost, which can be substituted to groundwater and address the question of the optimal switch time since the marginal water extraction cost increases with the depth of the water table. But it is often implicitly assumed that this alternate water source is exogenous like seawater desalinization, water im-

port or even new water sparing irrigation techniques. This means, in other word, that the switch to one of these alternate technique has no direct influence on the dynamic of the water table. This is typically not the case of RWH, especially in dry areas, since rainwater largely contributes to replenishment of the aquifer.

We deal therefore with an atypical conjunctive use system. In fact in the line of Gemma and Tsur [10], the term conjunctive signifies “that the ground and surface water sources are two components of one system and should be managed as such”. This clearly means that the dynamics of our two sources of water must be analyzed simultaneously which is the case in our paper. But, the second source is, here, a special kind of a surface water because it is the rainwater which can be harvested by the existing capacity. From that point of view, especially in dry areas,, RWH affords an alternative to groundwater use while in this conjunctive use literature (see for instance Tsur and Graham-Tomasi [38], Knapp and Olson [18] or Chakravorty and Umetsu [4]) groundwater is more viewed as an additional resource which insures against fluctuations in the amount of surface water or which helps in the organization of the production along a river. In our view, RWH is more considered as a substitute to pumping but the two techniques use the same resource since the former diminishes the natural recharge of the aquifer.

Our purpose, in this paper, is therefore to outline a trade-off between groundwater pumping and the collection of rainwater through investments in harvesting capital and, since both uses the same resource, to evaluate the effect of this practice on the level of the water table. We notably show that even if this two water harvesting techniques are pure substitutes, the development of RWH conducts in the long run to a depletion of the water table.

This puzzling result is not even based on strategic dynamic externalities between water users since we deliberately choose a social planner approach and therefore do not enter the debate around the Gisser Sanchez Effect<sup>1</sup>. This result comes from the existence of both a short and long-run effect directly issued from the implementation of an investment strategy. If we start with the last one , we can say that as soon as a quantity of capital is used to collect rainwater then a sustainable principle induces a reduction of the ground water use in the same proportion since the total withdrawn must be less then the rate on replenishment. But in the short run, when an investment occurs, the social planner has an incentive to postpone this adjustment in order to benefit from additional amount of water especially in the constitution of a harvesting capacity is not to costly. These gains are, of course, transitory since the cost of water extraction increases, in this case, across time. But this short term effect contributes to an additional depletion on the resource and therefore to a lower steady state equilibrium. Thus, the collection of rainwater does not allow really to

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<sup>1</sup>The paper of Gisser and Sanchez [11] gives rise to a large debate on the necessity to regulate private groundwater use in order to obtain the social planer associates solution (see Negri [25] Provencher [30] Provencher and Burt [31], Santiago-Rubio [35] and Rubio-Casino [33] among others)

spare groundwater.

In order to illustrate this point, we proceed in three steps. We first introduce a rather general model of water extraction in order to characterize the optimal harvesting strategy and to identify the steady state. In a second step, we allow RWH. This gives us the opportunity, by starting with steady state trajectory without RWH, to illustrate at least intuitively our argument by showing that an optimal planner who is willing to invest in the RWH technique has also an incentive to postpone the adjustment of her quantity of harvested groundwater and therefore induces a long term effect. But this argument is only obtain by considering deviations from a steady state trajectory without investment. This is why we verify, in a third step, this intuition within an optimal control model in which the social planner has the ability to choose both the groundwater extraction level on the investment in a RWH capacity.

The outline of this paper is as follows. Section 2 introduces the model by emphasizing all mathematical notations and the convenient assumptions for our model. Section 3 is dedicated to study of the optimal groundwater extraction within this setting in order to characterize both the dynamic and the steady state. In section 4 we introduce RWH, illustrate the idea that the planner has to deviate from the previous steady state and provide some intuition of the long run effect of this deviation. Section 5 analyzes the optimal behavior under both the groundwater extraction and the investment in a RWH capacity and concludes to a depletion of the resource with respect to the case without RWH. Finally, a brief discussion and concluding remarks are offered in section 6. All the proofs are relegated to an appendix.

## 1 The model

We start from a dynamic, continuous time model of groundwater management for an aquifer with a constant and natural<sup>2</sup> recharge  $R$ . The upper-surface of groundwater is called the water table which can rise in response to natural recharge and fall because of seasonally dry weather, drought or water pumping. Therefore, water table levels in aquifers represent the combined effects of the rate of recharge and the rate of discharge and, consequently, the amount of available water. To this end, the measurement of the depth of the aquifer is relevant for groundwater management. By the way, assume that the depth measuring at period  $t$  is  $d(t)$ . Obviously, if the water table reaches its upper limit, we can consider that the aquifer is full and we set  $d(t) = 0$ . At the opposite, if the aquifer is totally empty then the depth reaches its maximum level denoted by  $\bar{d}$ . Moreover, we assume the aquifer is characterized by a flat bottom and perpendicular sides. Therefore, the level of the water table is

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<sup>2</sup>A *natural recharge* results from snowmelt, precipitation or storm runoff. In this model, we rule out *artificial recharge*, i.e. the use of water coming from other sources to replenish the aquifer

the same in each point of the aquifer. Considering that the water table is shallower with the natural recharge  $R$  and deeper with the extraction  $w_g(t)$ , the dynamics of the water table across the time is normally given by  $\dot{d}(t) = w_g(t) - R$ .

In addition, we want to take into account the idea that water users have also the ability to capture directly a part of the natural recharge  $R$  by developing a collecting capacity of water harvesting such as rainwater tanks. We denote by  $w_s(t) \leq R$  the quantity that is directly withdrawn. However, harvesting rainwater is a way to stop the rainwater from hitting the ground and, therefore, replenishing the aquifer. Thus, the recharge that reaches the ground is reduced by the amount  $R - w_s(t)$  and the water depth dynamic is now given by :

$$\dot{d}(t) = w_g(t) - (R - w_s(t)) \quad (1)$$

Then, the total amount of water  $w = w_s + w_g$  which is obtained by harvesting and by pumping is affected to alternative uses. In fact, both factors can be used as two perfect substitutable inputs. Thus, we assume that a combination of these production factors provide a set of services which are measured by a social benefit function  $F(w)$  which behaves like a standard production function. More formally, we assume that :

**Assumption 1** *The social benefit of the use of water are measured by a  $C^\infty$  function  $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  which satisfies (i)  $F'(w) > 0$ , (ii)  $F''(w) < 0$ , (iii)  $F(0) = 0$ , (iv)  $\lim_{w \rightarrow 0} F'(w) = +\infty$  and (v)  $\lim_{w \rightarrow +\infty} F'(w) = 0$*

The use of this amount  $w$  of water is however not free of charge. Groundwater withdrawals induce some pumping costs while any harvesting capital requires some investment in order to develop and to maintain a harvesting capacity of  $w_s$ .

To be more precise, we assume as usually in this literature ([15] and [35]) that the resource exploitation involves a cost  $C(w_g, d)$  depending on the amount that is pumped and on the depth of the water table. Thus, we are able to capture two basic principles. The first one stems from the fact that, at lower water tables, it is more costly to extract water because the resource must be pumped farther distances. Therefore, the marginal cost of pumping a unit of water is increasing with the depth of the water table (*i.e.*  $\partial_{w_g, d}^2 C(w_g, d) > 0$ ). Then, the second principle is related to the dynamic of the model. Actually, the use of an additional unit of water, at a given period of time, decreases the water table and, according to this new level, rises all the future extraction costs : this is a crucial point while this natural resource has an economic value.

Beyond this two basic principles, we also assume decreasing return to scale in the sense that :

- the marginal cost of extracting water is infinitely increasing when the quantity of water tends to infinite. We however do not assume that the marginal extraction cost of the first unit of water is always zero because, of course, the depth of the water table surely matters. We simply require that this marginal cost is bounded from above, this bound being perhaps very large when the aquifer is almost empty.
- the pumping cost are increasing with the depth of the water table at an increasing rate. But, when no water is taken the depth of the water table does not really matter in the cost function. This implies in particular, by abuse of notation, that  $\partial_d C(0, d) = 0$  since  $C(0, d)$  is assumed to be constant. Furthermore, this assumption follows the usual total pumping cost which is linear in pumping lift.
- this cost function is strictly convex or, in other words, that the Hessian of this function is a negative definite matrix.

More formally, we say that :

**Assumption 2** *The groundwater extraction costs are given by a  $C^\infty$  function  $C : \mathbb{R}_+ \times [0, \bar{d}] \rightarrow \mathbb{R}_+$*

*(i)  $\partial_{w_g} C(w_g, d) > 0$ ,  $\partial_{w_g, w_g}^2 C(w_g, d) > 0$ ,  $\forall d$ ,  $\partial_{w_g} C(0, d) < K$  bounded from above and  $\lim_{w_g \rightarrow \infty} \partial_{w_g} C(w_g, d) = +\infty$*

*(ii)  $\forall w_g > 0$ ,  $\partial_d C(w_g, d) > 0$ ,  $\partial_{d, d}^2 C(w_g, d) > 0$  and  $\partial_d C(0, d) = 0$*

*(iii)  $\partial_{w_g, w_g}^2 C(w_g, d) \cdot \partial_{d, d}^2 C(w_g, d) - \left( \partial_{w_g, d}^2 C(w_g, d) \right)^2 > 0$*

*(iv)  $\partial_{w_g, d}^2 C(w_g, d) > 0$*

In addition, we also allow for water harvesting, a new aspect within this literature. But, this technology requires an investment in order to build and to maintain this irrigation capacity. To keep the model as simple as possible, we however assume, as usual in a standard growth model, that this capacity can be adjusted instantaneously. This means, in other words, that this capacity coincides with the level of harvested water and that its dynamics takes into account not only the investment  $I$  which is realized each period at some cost  $\Theta(I)$  but also the depreciation of this capital which is measured by the function  $\delta(w_s)$ . The dynamics of the capital stock across time is therefore given by the relation :

$$\dot{w}_s(t) = I(t) - \delta(w_s(t))$$

Instead of taking a linear depreciation function, we prefer to keep a more generalized form which fits with some standard assumptions. Effectively, we assume that



the depreciation function is an increasing strictly convex function. The depreciation is increasing with the amount of capital at an accelerated rate. Moreover, when there is no capital, there is obviously no depreciation and, we assume furthermore that the first unit of capital does not imply any depreciation.

In parallel, the underlying costs of adjustment that must be paid out of operating profits are increasing and strictly convex in  $I$  as in Abel and Eberly [1]. Like for the depreciation function, we assume that there is no adjustment cost for non-existing investment and the first unit of investment does not involve any charge.

More formally, we set that :

**Assumption 3** *The water harvesting technique is characterized by a  $C^\infty$  investment cost  $\Theta : \mathbb{R} \rightarrow \mathbb{R}_+$  and a  $C^\infty$  depreciation function  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  which respectively satisfy :*

- (i)  $\forall I > 0, \Theta'(I) > 0, \Theta''(I) > 0$  and  $\Theta(0) = \Theta'(0) = 0$
- (ii)  $\forall w_s > 0, \delta'(w_s) > 0, \delta''(w_s) > 0$  and  $\delta(0) = \delta'(0) = 0$

Before going further, we introduce the two following assumptions :

- If the aquifer is full ( $d = 0$ ) and only the natural recharge is consumed ( $w_g = R$ ), there is always an incentive, at least marginally, to pump an additional quantity of water even by taking into account the increase in future extraction cost induced by a change in the water table.
- At the opposite, the marginal cost of extracting the last unit of water when the aquifer is empty ( $d = \bar{d}$ ) is very high, at least higher than the marginal productivity of the recharge. Furthermore, we assume that marginal cost of investment defining by the product of the marginal adjustment cost with the sum of the discount rate and the depreciation rate when we have already invested to capture the entire recharge  $R$  is larger than the marginal cost of extracting the last unit of water when the aquifer is empty. Beyond the mathematical convenience of this assumption, it is quite credible that the marginal cost of investment is very large when we have already collected all the recharge compared to the cost of one unit of groundwater when the aquifer is empty whereas we do not pump any quantity.

These assumption can be written as :

**Assumption 4** *Let us assume that*

- (i)  $F'(R) - \partial_{w_g} C(R; 0) - \frac{1}{\rho} \partial_d C(R; 0) > 0$
- (ii)  $\Theta'(\delta(R))(\rho + \delta'(R)) > \partial_{w_g} C(0, \bar{d}) > F'(R)$

## 2 The standard groundwater management model

The social planner will choose the optimal extraction path maximizing the total present values of social welfare. Formally, the social planner's problem is given by :

$$\begin{aligned} \max_{w_g(t) \in \Omega_1(d(t))} \mathcal{J}_1(w_g(t), d(t)) &= \int_0^\infty [F(w_g(t)) - C(w_g(t), d(t))] e^{-\rho t} dt \quad (2) \\ \begin{cases} \dot{d}(t) = w_g(t) - R, & d(0) = d_0, & d(\infty) \text{ free} \\ w_g(t) \geq 0, & d(t) \geq 0, & \bar{d} - d(t) \geq 0 \end{cases} \end{aligned}$$

This is typically an autonomous optimal control problem with a mixed and two pure state constraints. Moreover, by using the so-called indirect approach<sup>3</sup> we know that the set of admissible control values is given by :

$$\Omega_1(d(t)) = \{w_g \in \mathbb{R}_+ : w_g \geq R \text{ if } d(t) = 0, w_g \leq R \text{ if } d(t) = \bar{d}\}$$

The associated current-value Hamiltonian with co-state variable  $p(t)$  is defined by

$$\mathcal{H}_1(w_g(t), d(t), p(t)) = [F(w_g(t)) - C(w_g(t), d(t))] + p(t) \cdot (w_g(t) - R)$$

It is immediate that  $\mathcal{H}_1$  is strictly concave with respect to  $w_g$ . Moreover, since the admissible control set is convex, we can even expect the following properties.

**LEMMA 1** *Under Assumptions (1) and (2), this optimal control problem*  
*(i) is regular (the  $\arg \max_{w_g \in \Omega_1(d)} \mathcal{H}_1(w_g, d, p)$  is a singleton for all  $d$  and  $p$ ).*  
*(ii) admits an optimal control path  $\tilde{w}_g(t)$  which is continuous and strictly positive.*  
*(iii) satisfies the constraint qualification.*  
*(iv) has the property that the co-state variable  $\tilde{p}(t)$  and the Hamiltonian along the optimal path are continuous.*

Hence, from lemma 1, we know that the mixed constraint  $w_g(t) \geq 0$  can be forgotten. Thus, we can now introduce the following Lagrangian affiliated to the program (2) :

$$\mathcal{L}_1(w_g(t), d(t), p(t), q_1(t), q_2(t)) = \mathcal{H}_1(w_g(t), d(t), p(t)) + q_1(t) \cdot d(t) + q_2(t) \cdot (\bar{d} - d(t))$$

where  $q_1(t)$  and  $q_2(t)$  are the multiplicative associated to the two pure state constraints. We can now claim that a solution to our problem satisfies the following

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<sup>3</sup>The indirect approach consists of adjoining a function instead of the pure state constraints. For more details, see Seierstad and Sydsaeter [36] or Grass and *al.* [12]

*Almost Necessary Conditions* (see Seierstad and Sydsaeter [36] theorem 9 p.381 and note 6 p.374 or Grass and *al.* [12] theorem 3.60 p.149)

$$F'(w_g(t)) - \partial_{w_g} C(w(t), d(t)) + p(t) = 0 \quad (3)$$

$$\dot{p}(t) = \rho p(t) + \partial_d C(w_g(t), d(t)) - q_1(t) + q_2(t) \quad (4)$$

with the complementary slackness conditions :

$$\begin{cases} q_1(t) \geq 0 & q_1(t) \cdot d(t) \geq 0 \\ q_2(t) \geq 0 & q_2(t) \cdot (\bar{d} - d(t)) \geq 0 \end{cases}$$

Before going further, it is interesting in noticing that the two pure state constraints are never binding.

**LEMMA 2** *When an optimal control is at work, it is impossible to find a period of time  $]t_0, t_1[$  for which*

- (i) *the aquifer is totally full, i.e.  $\tilde{d}(t) = 0$ .*
- (ii) *the aquifer is totally empty, i.e.  $\tilde{d}(t) = \bar{d}$*

If the Hamiltonian  $\mathcal{H}_1(w_g, d, p)$  is strictly concave in  $(w_g, d)$  and the different constraints are quasi-concave in these variables, we can even say by using Mangasarian type sufficient conditions (see Seierstad and Sydsaeter [36] theorem 11 p.385) that, in our case, the optimal solution satisfies the following proposition.

**Proposition 1** *Any triple  $(w_g^*(t), d^*(t), p^*(t))$  of functions which satisfies*

$$F'(w_g^*(t)) - \partial_w C(w_g^*(t), d^*(t)) + p^*(t) = 0 \quad (5)$$

$$\dot{p}^*(t) = \rho p^*(t) + \partial_d C_d(w_g^*(t), d^*(t)) \quad (6)$$

$$\dot{d}^*(t) = w_g^*(t) - R \quad (7)$$

$$\lim_{t \rightarrow \infty} p^*(t) (d(t) - d^*(t)) \geq 0 \text{ for all admissible } d(t) \quad (8)$$

*is the unique optimal solution.*

At that point, the reader may perhaps be surprised by our treatment of the shadow price  $p^*(t)$  compared to the rest of the literature ([15] and [35]). Usually, this marginal user cost is equal to the royalties, at the optimum. However, this follows directly from the fact that we use as a state variable the depth of the aquifer instead of its height. From that point of view,  $p^*(t)$  does not measure the long run benefit from a marginal increase of the water table along the optimal path but measures exactly the opposite since an increase in the depth induces a decline in the water table. This means, in other words, that when we move to a representation into

the state-control space, we will come back to a standard representation. In fact, in this space, the dynamics of our system can be represented by :

$$A(w_g(t), d(t)) \cdot \begin{bmatrix} \dot{w}_g(t) \\ \dot{d}(t) \end{bmatrix} = b(w_g(t), d(t))$$

with

$$A(w_g(t), d(t)) = \begin{bmatrix} -F''(w_g(t)) + \partial_{w_g, w_g}^2 C(w_g(t), d(t)) & \partial_{w, d}^2 C(w_g(t), d(t)) \\ 0 & 1 \end{bmatrix}$$

and  $b(w_g(t), d(t)) = \begin{bmatrix} \rho(-F'(w_g(t)) + \partial_w C(w_g(t), d(t))) + \partial_d C_d(w_g(t), d(t)) \\ w_g(t) - R \end{bmatrix}$

Hence :

$$\begin{cases} \dot{w}_g(t) = \frac{\rho(F'(w_g(t)) - \partial_w C(w_g(t), d(t))) - \partial_d C(w_g(t), d(t))}{F''(w_g(t)) - \partial_{w_g, w_g}^2 C(w_g(t), d(t))} + \frac{\partial_{w_g, d}^2 C(w_g(t), d(t)) \cdot (w_g(t) - R)}{F''(w_g(t)) - \partial_{w_g, w_g}^2 C(w_g(t), d(t))} = W(w_g(t), d(t)) \\ \dot{d}(t) = w_g(t) - R \end{cases}$$

which can be a rather complicated dynamics. But the definition of the steady state remains quite simple : it is given by  $b(w_g^*, d^*) = 0$ . Moreover the matrix  $D$  which describes the first order approximation of this system in the neighborhood of this point, is rather tractable. It is after some computations given by :

$$\begin{aligned} D &= \begin{bmatrix} \partial_{w_g} W(w_g, d)|_{(w_g^*, d^*)} & \partial_d W(w_g, d)|_{(w_g^*, d^*)} \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\rho(F'' - \partial_{w_g, w_g}^2 C)}{F'' - \partial_{w_g, w_g}^2 C} \Big|_{(w_g^*, d^*)} & -\frac{\rho \partial_{w_g, d}^2 C + \partial_{d, d}^2 C}{F'' - \partial_{w_g, w_g}^2 C} \Big|_{(w_g^*, d^*)} \\ 1 & 0 \end{bmatrix} \end{aligned}$$

This is why we can say that :

**Proposition 2** *If we concentrate our attention on the steady state, we observe that*  
*(i) This point is unique and is given by  $w_g^* = R$  and  $d^* \in ]0, \bar{d}[$  which satisfies*

$$F'(R) - \partial_w C(R, d^*) - \frac{1}{\rho} \partial_d C_d(R, d^*) = 0$$

*(ii) at least locally (i.e. at a neighborhood of the steady state), this two-dimensional system admits a unique saddle path which converges to the steady state*

To conclude, this model is suitable to determine to what extent groundwater can be withdrawn without compromising the resource for the future. On this basis, any deviation can be analyzed when a new water source becomes accessible.

### 3 Potential variations of the basic steady state

The previous section presents a model where groundwater is used as a single source of irrigation. Now, we introduce the possibility to harvest a quantity  $w_s(t)$  of rainwater at the period  $t$  in order to extend freshwater source. Thereby, rainwater and groundwater will be used simultaneously as substitutable inputs in the production function such that  $F(w_g(t) + w_s(t))$ .

The collection of rainwater harvesting stems from the amount of investment made by water users. For convenience, we assume for the moment that there is no capital depreciation. Thus, if investment is incitive we can expect that a part of groundwater will be substituted by an equivalent quantity of rainwater and, without long-run effect, the depth of the water table will not be affected. However, we are going to show that this intuition is wrong or, in other words, that there exists a long-run effect resulting from the investment and, thus, influencing the level of the water table.

In order to illustrate this mechanism, we are going to use some principles of the calculus of variations. We already know the optimal extraction path highlighting in the basic groundwater model in which  $w_g^*(t) = R$ . Then, we analyze a deviation from this optimal trajectory by allowing to invest a constant amount  $\Delta I$  during a finite period  $t \leq t_I$ . After that, no more investment will be made. Thus, the variation of the investment  $I(t)$  is written such that :

$$I(t) = \begin{cases} \Delta I & \forall t \leq t_I \\ 0 & \forall t > t_I \end{cases}$$

Since there is no capital depreciation, the variation of rainwater harvesting comes directly from the investment. Thus, we set :

$$\dot{w}_s(t) = I(t) \tag{9}$$

Moreover, we can compute the quantity of stored rainwater by adding all water collected at each period :

$$w_s(t, I(t)) = w_s(0) + \int_0^t \dot{w}_s(t) dt \tag{10}$$

$$= \begin{cases} \Delta I \cdot t & \forall t \leq t_I \\ \Delta I \cdot t_I & \forall t > t_I \end{cases} \tag{11}$$

Since this technology has just been introduced, it is obvious that the initial condition is  $w_s(0) = 0$ .

Up to now, the social planner has two resources which can be combined in various ways. In fact, it is intuitive to observe that he faces with two strategies : either he continues to extract exactly the recharge from the ground in addition to a quantity

of rainwater or he chooses to adjust his withdrawals to the amount of the harvested rainwater. However, given that  $w_g^*(t) = R$ , then the dynamics (1) becomes  $\dot{d}(t) = w_s(t)$ . Therefore, under the assumption that investment is incitive then the depth of the water table will be increasing across time, with  $w_s(t) > 0$ . This means that the aquifer will become deeper until reaching its bottom  $\bar{d}$ . Therefore, a strategy without adjustment is not admissible.

Given this observation, let  $t_{w_g}$  the period from which a strategy of substitution is implemented. Obviously, the adjustment will also depend on the investment switch time. Thus, the amount of groundwater that is pumped can be written as following :

$$w_g(t) = \begin{cases} R & \forall t \leq t_{w_g} \\ R - \Delta I \cdot t & \forall t_{w_g} < t \leq t_I \\ R - \Delta I \cdot t_I & \forall t > t_I \geq t_{w_g} \end{cases} \quad (12)$$

Hence, we can compute the level of the water table :

$$\begin{aligned} d(t, w_g(t), I(t)) &= d(0) + \int_0^t \dot{d}(t) dt \\ &= \begin{cases} d_0 + \Delta I \cdot \frac{t^2}{2} & \forall t \leq t_{w_g} \\ d_0 + \Delta I \cdot \frac{t_{w_g}^2}{2} & \forall t > t_{w_g} \end{cases} \end{aligned} \quad (13)$$

We can observe that there is a long run effect  $\Delta I \cdot \frac{t_{w_g}^2}{2}$  when the substitution is not immediate.

Now, we can derive the social net benefit when such a variation is made. It is defined as the difference between the production function and the total cost which is the sum of the extraction cost and the investment adjustment cost. Thus, the total present values of the social welfare is given by the following functional :

$$\begin{aligned} J(\Delta I, t_{w_g}, t_I) &= \int_0^{t_{w_g}} \left( F(R + \Delta I \cdot t) - C\left(R, d_0 + \Delta I \cdot \frac{t^2}{2}\right) - \Theta(\Delta I) \right) \exp^{-\rho t} dt \\ &+ \int_{t_{w_g}}^{t_I} \left( F(R) - C\left(R - \Delta I \cdot t, d_0 + \Delta I \cdot \frac{t_{w_g}^2}{2}\right) - \Theta(\Delta I) \right) \exp^{-\rho t} dt \\ &+ \int_{t_I}^{\infty} \left( F(R) - C\left(R - \Delta I \cdot t_I, d_0 + \Delta I \cdot \frac{t_{w_g}^2}{2}\right) - \Theta(0) \right) \exp^{-\rho t} dt \end{aligned} \quad (14)$$

Obviously, such a strategy will be implemented if it is profitable. In other words, investment must generate a higher profit even if there are additional costs. We can show that investing to store rainwater is a beneficial choice compared to using only groundwater for irrigation even if the adjustment between rainwater and groundwater is applied at the beginning.

*Remark 1 It is always interesting in investing even if we adjust the extraction at the beginning, i.e.  $t_{wg} = 0$ .*

$$\lim_{\Delta I \rightarrow 0^+} \frac{J(\Delta I, 0, t_I) - J(0, 0, 0)}{\Delta I} = \frac{(1 - \exp^{-\rho t_I})}{\rho} \left( \frac{1}{\rho} \partial_{w_g} C(R, d_0) - \underbrace{\Theta'(0)}_{=0} \right) > 0 \quad (15)$$

In effect, the actualized marginal pumping cost when we have already extracted the recharge is greater than the marginal cost of investment when we did not invest yet. Therefore, there is an interest in investing in this irrigation capacity.

However, we have already observed that a strategy of substitution must be implemented with investment. But, here is no reason so as not to postpone the time of adjustment. In fact, we can even demonstrate that there exists an interest in substituting both resources after a short period.

*Remark 2 The adjustment strategy is not optimal when it is applied at the beginning because it is a local minimum.*

$$\partial_{t_{wg}} J(\Delta I, t_{wg}, t_I) \Big|_{t_{wg}=0} = 0, \quad \partial_{t_{wg}, t_{wg}}^2 J(\Delta I, t_{wg}, t_I) \Big|_{t_{wg}=0} > 0 \quad (16)$$

Thus, during a short period, the production will temporarily increase.

As result, we have illustrate the idea that investment is a beneficial strategy but it must come with a substitution between rainwater and groundwater withdrawals in order to prevent from depleting the aquifer. However, beyond this substitution, various effects influence the structure of costs. Actually, the collection of rainwater implies that the natural recharge that reaches the ground is reduced by  $R - w_s(t)$ . Thus, additional costs are involved because water users are going to pump water at farther distance. Consequently, this mechanism incites to invest and to continue to extract an amount equal to the recharge during a short period. Thereby, when the substitution will be actual, there will remains a long-run effect that explains that the level of the water table will be deeper.

Obviously, this is only the intuition of the mechanism that is at work since we have looked at a rather peculiar deviation from the path which was given by the steady state without water harvesting. It remains now to establish this result more formally by looking at the optimal choice of a social planner which has the opportunity to pump groundwater and to develop some harvesting capacity. In fact, we will argue in two steps. We are first looking at the optimal path in order to highlight the idea, among others, that has always an incentive to develop a harvesting capacity and to maintain (when there are a natural depreciation of this capital) some investment across time. Then, we will move in a second step to the study of the new steady state that we completely characterize in order to evaluate the long term effect of water harvesting.

## 4 Groundwater extraction and investment

The social planner will now choose the optimal paths of groundwater and investment maximizing the total present values of the social welfare. This problem is very closed to the previous one. We simply add the ability to obtain a harvesting capacity by making a costly investment, this capacity being non negative and bounded by the natural recharge of the aquifer. Formally, the problem becomes :

$$\begin{aligned} & \max_{w(t), I(t)} \int_0^\infty (F(w_g(t) + w_s(t)) - C(w_g(t); d(t)) - \Theta(I(t))) e^{-\rho t} dt \quad (17) \\ & \text{subject to : } \begin{cases} \dot{d}(t) = w_g(t) + w_s(t) - R, \text{ with } d(0) = 0 \\ \dot{w}_s(t) = I(t) - \delta(w_s(t)) \text{ with } w_s(0) = 0 \\ w_g(t) \geq 0, d(t) \geq 0, \bar{d} - d(t) \geq 0 \\ w_s(t) \geq 0, R - w_s(t) \geq 0 \end{cases} \end{aligned}$$

We will denote by  $\mathcal{H}_2(w_g(t), I(t), w_g(t), d(t), (p_i(t))_{i=1}^2)$  the Hamiltonian associated to this program with  $p_1(t)$  and  $p_2(t)$  the two co-state variables related respectively to the dynamic of the aquifer and of the harvesting capacity.

$\mathcal{L}_2(w_g(t), I(t), w_g(t), d(t), (p_i(t))_{i=1}^2, (q_i(t))_{j=1}^5)$  stands for the associated Lagrangian where the  $q_i(t)$  are associated to the five constraints. If we now move to the study of this program, we first observe that :

**LEMMA 3** *The constraint qualification property is satisfied.*

From that point of view, we can say that  $(w_g(t), I(t), w_s(t), d(t))$  is an optimal solution path with only finitely many time junctions, if there exists piecewise absolutely continuous functions  $(p_i(t))_{i=1}^2$ , and  $(q_i(t))_{j=1}^5 \geq 0$  as well as a vector  $(\eta_{i,\bar{t}})_{i=1}^4$  at each junction point  $\bar{t}$  which satisfies the following conditions:

$$\begin{cases} F'(w_g(t) + w_s(t)) - \partial_{w_g} C(w_g(t), d(t)) + p_1(t) + q_1(t) = 0 \\ -\Theta'(I(t)) + p_2(t) = 0 \\ \dot{p}_1(t) = \rho p_1(t) + \partial_{d_t} C(w_g(t), d(t)) - q_2(t) + q_3(t) \\ \dot{p}_2(t) = p_2(t) (\rho + \delta'(w_s(t))) - F'(w_g(t) + w_s(t)) - p_1(t) - q_4(t) + q_5(t) \\ q_1(t) \cdot w_g(t) = 0, \quad q_2(t) \cdot d(t) = 0, \quad q_3(t) \cdot (\bar{d} - d(t)) = 0 \\ q_4(t) \cdot w_s(t) = 0 \text{ and } q_5(t) \cdot (R - w_s(t)) = 0 \end{cases}$$

and at each junction point  $\bar{t}$ , there exists  $(\eta_{i,\bar{t}})_{i=1}^4 \geq 0$  with the property that :

$$\begin{cases} p_1(\bar{t}^+) = p_1(\bar{t}^-) - \eta_{1,\bar{t}} + \eta_{2,\bar{t}} \\ p_2(\bar{t}^+) = p_2(\bar{t}^-) - \eta_{3,\bar{t}} + \eta_{4,\bar{t}} \end{cases} \quad \text{and} \quad \begin{cases} \eta_{1,\bar{t}} \cdot d(\bar{t}) = 0, \eta_{2,\bar{t}} \cdot (\bar{d} - d(\bar{t})) = 0 \\ \eta_{3,\bar{t}} \cdot w_s(\bar{t}) = 0, \eta_{4,\bar{t}} \cdot (R - w_s(\bar{t})) = 0 \end{cases}$$

Moreover, we can even say that



**LEMMA 4** *Under our assumptions, any path  $(\tilde{w}_g(t), \tilde{I}(t), \tilde{w}_s(t), \tilde{d}(t))$  which satisfies the previous conditions is an optimal path providing that :*

$$\lim_{t \rightarrow +\infty} p_1(t) (d(t) - \tilde{d}(t)) + p_2(t) (w_s(t) - \tilde{w}_s(t)) = 0$$

for all admissible  $(w_s(t), d(t))$

In fact, a more in-depth perusal of such a program shows that the optimal path has several characteristics that are relevant to describe it. Proposition 3 presents those properties.

**Proposition 3** *The optimal path has several interesting properties. We can note that :*

- (i) *Water is always used since  $w_g(t) + w_s(t) > 0$ .*
- (ii) *The harvesting capacity is always strictly positive i.e  $w_s(t) > 0$ .*
- (iii) *When water is pumped ( $w_g(t) > 0$ ) there is always a strictly positive investment ( $I(t) > 0$ ).*
- (iv) *When no water is pumped ( $w_g(t) = 0$ ), the harvesting capacity does not reach the recharge ( $w_s(t) < R$ ).*
- (v) *If the aquifer is either full or empty for a while (i.e.  $\exists ]t_0, t_1[, \forall t, d(t) = 0, \bar{d}$ ) the harvesting capacity changes across time (i.e  $\dot{w}_s(t) \neq 0$ ).*

The *first point* insures that we will always observe an active productive sector using at least one of the two sources of water. In fact, if we can observe that when no rainwater is harvested then, from assumption 4, this observation means that water users do not pay out any investment cost and, we obtain a situation equivalent to the scenario using groundwater as a single input where extraction occurs. Besides, the *second point* goes into detail in the sense that the optimal path is characterized by the implementation of the harvesting technology. This property is quite intuitive in accordance with what we have learnt in the previous section. The *third point* goes a step further. It tells us that as soon as groundwater is extracted, there is an investment in a harvesting capacity. This also insures that a part of the deteriorated capital is replaced in order to maintain the technology. The last properties provide some intuition on the steady state. According to the *fourth point*, if there is no withdrawal, then it is not optimal to maintain a harvesting capacity at the level of the recharge. Since in the long run, all the recharge is used this suggests that water is extracted in the steady state. Finally, by the *fifth point*, if the aquifer is either full or empty along the optimal path there must be some adjustments in the harvesting capacity. In other words it is impossible to reach a steady state when the aquifer cannot reach its boundaries.

## 5 The long term effect on the aquifer

Let us now show that the introduction of water harvesting techniques has a long run effect on the aquifer : it induces a lower equilibrium level of the natural resource. This long run stationary equilibrium typically solves the set of equations (18) with  $\dot{p}_1(t) = \dot{p}_2(t) = 0$ , the set (18) and has the property that  $\dot{w}_s(t) = \dot{d}(t) = 0$ . We therefore know that the harvesting capacity must be equal to the recharge net of the used groundwater (i.e.  $w_s^* = R - w_g^*$ ), the investment must compensate the depreciation of this capacity (i.e.  $I^* = \delta(R - w_g^*)$ ) and the shadow price of this capacity reflects the marginal cost of this long term investment (i.e.  $p_2^* = \Theta'(\delta(R - w_g^*))$ ). We can even observe that the shadow price associated to the aquifer depth is given by  $p_1^* = -\frac{1}{\rho} (\partial_d C(w_g^*, d^*) - q_2^* + q_3^*)$ .

After some rearrangements, we can therefore say that any couple  $(w_g^*, d^*)$  given by a level of groundwater extraction and a water table depth induces a stationary equilibrium if we can find a vector a Lagrangian multiplicatives  $(q_i^*)_{i=1}^5$  which satisfies:

$$\begin{cases} F'(R) - \partial_{w_g} C(w_g^*, d^*) - \frac{1}{\rho} \partial_d C(w_g^*, d^*) = \frac{1}{\rho} (q_3^* - q_2^*) - q_1^* \\ \Theta'(\delta(R - w_g^*)) (\rho + \delta'(R - w_g^*)) - \partial_{w_g} C(w_g^*, d^*) = q_4^* - q_5^* - q_1^* \\ q_1^* \cdot w_g^* = 0, q_2^* \cdot d^* = 0, q_3^* \cdot (d - d^*) = 0, q_4^* \cdot (R - w_g^*) = 0 \text{ and } q_5^* \cdot w_g^* = 0 \end{cases}$$

The set of stationary equilibria can therefore be quite large. But if we come back to proposition 3 (iv) and (v) and in particular to the interpretation that we gave, we can expect that :

**LEMMA 5** *Any stationary equilibrium is an interior point, i.e.  $w_g^*, I^* > 0$ ,  $d^* \in ]0, \bar{d}[$  and  $w_s^* \in ]0, R[$ , or in other words  $(q_i^*)_{i=1}^5 = 0$*

This preliminary result is quite interesting since it reduces the search of the stationary equilibrium to the set of all  $(w_g^*, d^*)$  in  $]0, R[ \times ]0, \bar{d}[$  which solve

$$\begin{cases} \Phi_1(w_g^*, d^*) = -F'(R) + \partial_{w_g} C(w_g^*, d^*) + \frac{1}{\rho} \partial_d C(w_g^*, d^*) = 0 \\ \Phi_2(w_g^*, d^*) = \underbrace{\Theta'(\delta(R - w_g^*)) (\rho + \delta'(R - w_g^*))}_{A(w_g^*)} - \partial_{w_g} C(w_g^*, d^*) = 0 \end{cases}$$

and under our assumptions we can even assert that :

**Proposition 4** *There exists a unique interior stationary equilibrium since the previous system of equations has a unique solution in  $]0, R[ \times ]0, \bar{d}[$*

The intuition beyond this result is quite simple. In fact when we apply the implicit function theorem respectively to  $\Phi_1$  and  $\Phi_2$  we obtain

$$\left. \frac{dd}{dw_g} \right|_{\Phi_1=0} = - \frac{\partial_{w_g, w_g}^2 C(w_g, d) + \frac{1}{\rho} \partial_{d, w_g}^2 C(w_g, d)}{\partial_{w_g, d}^2 C(w_g, d) + \frac{1}{\rho} \partial_{d, d}^2 C(w_g, d)} < 0$$

and

$$\left. \frac{dd}{dw_g} \right|_{\Phi_2=0} = \frac{\frac{dA}{dw_g} - \partial_{w_g, w_g}^2 C(w_g, d)}{\partial_{w_g, d}^2 C(w_g, d)} < 0$$

with  $\frac{dA}{dw_g} = -\Theta''(\delta(R-w_g^*)) \times \delta'(R-w_g^*) \times (\rho + \delta'(R-w_g^*)) - \delta''(R-w_g^*) \times \Theta'(\delta(R-w_g^*)) < 0$

So (see figure 1) if the relative slopes of this two curves are well-behaved and the curves  $(\Phi_1(w_g, d) = 0)$  and  $(\Phi_2(w_g, d) = 0)$  cut the border of the box  $[0, R] \times [0, \bar{d}]$  respectively at (2)-(4) and (3)-(1), we obtain the existence and uniqueness of a stationary equilibrium. The first result is induced by the study of the Jacobian of  $\Phi$  and requires assumptions 2 and 3 on pumping cost and investment while the second result follows from boundary behaviors and therefore requires assumption 4.

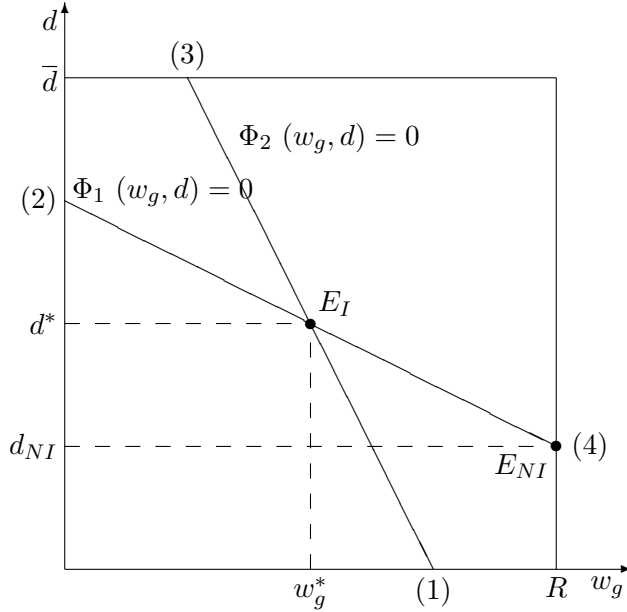


Figure 1: The long run equilibrium

If we look at this picture, we can even go a step further. In fact, let us remember that the steady state of a our groundwater extraction model without investment can be obtained by solving  $\Phi_1(R, d_{NI}) = 0$ . It follows that the depth of the aquifer is given by the intersection of  $(\Phi_1(w_g, d) = 0)$  and the right border  $w_g = R$ . We can therefore easily compare the two situations.

**Proposition 5** *By allowing water harvesting, we observe that :*

*(i) there is in the long run a pure substitution between the use of ground and harvested water; i.e.  $w_g^* + w_s^* = R$*

*(ii) but there is in the short run a over-exploitation of the aquifer so that in the long run the aquifer will be deeper, i.e.  $d_{NI} < d^*$*

## Conclusion

This paper derives the standard groundwater extraction model in order to introduce the opportunity for accessing a new source of water, i.e. rainwater, through investment in capital. Rainwater Harvesting is a technology requiring investment to build and to maintain an irrigation capacity that can be used jointly with groundwater. The derived conclusion of this model leads us to observe that the level of the aquifer, at the steady state, will be deeper in the presence of this irrigation capacity. This steady state results from the trade-off driving marginal costs of exploiting the aquifer and investing.

However, to isolate the effect of rainwater harvesting on groundwater extraction as well as on the level of the aquifer depth, we consider the simplest possible dynamic setting with (i) a simple “bathtub” aquifer, i.e. a flat bottom with parallel sides, (ii) the social planner approach (iii) complete information on hydrological characteristics (iv) no uncertainty on capital.

These simplifications call for future extensions. Namely, in line with the literature relaxing some Gisser-Sanchez assumptions, it could be interesting to incorporate more accurate depiction of groundwater hydrology and rainwater variability. For instance, Brozovic and *al.* [3] or Saak and Peterson [34] integrate spatially variable feature such as the speed of lateral flow or differences in the elevation of bottom. Thus, within our framework, one can expect that the consideration of a two-cell aquifer where the elevation of bottom differs across location may impact our result through the trade-off based on marginal costs.

Another more scrupulous characterization could lead us to incorporate uncertainties about rainfall variability following, for instance, Fisher and Rubio [8] who model water resources as a stochastic process and focus on the determination of long-run water storage capacity. Actually, the failure to include uncertainty can lead to costly errors. In other words, by reckoning random capital in order to capture fluctuations in precipitations, one can expect that the level of the aquifer depth in the steady state will dependent on risk behavior as well as the uncertainty level.

Thus, various refinements can be stemmed from this model allowing for more detailed approach. Nevertheless, expected results should be relatively similar with our primary finding, i.e. the impact of the aquifer. Hence, one can wonder about the meaning of this result with respect to the principle of sustainable development.

Actually, groundwater insures also the maintenance of ecosystem health which gives it a conservation value. In other words, one can wonder whether the implementation of this technology does not challenge the sustainable level of groundwater which insures all its functions.

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## Appendix

### Proof of lemma 1

Point (i) :  $\forall d, p$ ,  $\arg \max_{w_g \in \Omega_1(d)} \mathcal{H}_1(w_g, d, p)$  is a singleton.

Let us first observe that  $\partial_{w_g, w_g}^2 \mathcal{H}_1(w_g, d, p) = F''(w_g) - \partial_{w_g, w_g}^2 C(w_g, d) < 0$ . The Hamiltonian is therefore strictly concave in  $w_g$ . From that point of view, the result is obvious for  $d = \bar{d}$  because we maximize a continuous strictly concave function on  $\Omega(\bar{d}) = \{w_g \in \mathbb{R} : 0 \leq w_g \leq R\}$  a non-empty, compact and convex set. But when  $d \in ]0, \bar{d}[$ ,  $\Omega_1(d) = \mathbb{R}_+$  is no longer bounded from above. We nevertheless know from assumptions (1) and (2) that :

$$\lim_{w_g \rightarrow +\infty} \partial_{w_g} \mathcal{H}_1(w_g, d, p) = \lim_{w_g \rightarrow +\infty} (F'(w_g) - \partial_{w_g} C(w_g, d) + p) = -\infty.$$

So if we impose a finite bound  $K_n$  and push this bound to  $+\infty$ , it is impossible to construct a sequence of maxima  $\hat{w}_g^n \rightarrow +\infty$  since, after some rank  $N$ , the first order necessary condition will not be met. This unbounded problem has therefore a solution and even a unique one since  $\mathcal{H}_1$  is strictly concave and  $\Omega_1(d)$  is convex. Finally if  $d = 0$ , then  $\Omega_1(0) = [R, +\infty[$  and we can argue as previously as long as  $R < K_n$ .

### Proof of proposition 1

It remains to verify that the Hamiltonian  $\mathcal{H}_1(w_g, d, p)$  is strictly concave in  $(w_g, d)$  and that the different constraints are quasi-concave in these variables. This last condition is always satisfied since our constraints are linear. So let us now compute for each  $p$ , the Hessian of  $\mathcal{H}_1(w_g, d, p)$ . We obtain :

$$\partial^2 \mathcal{H}_1 = \begin{bmatrix} F''' - \partial_{w_g, w_g}^2 C & -\partial_{w_g, d}^2 C \\ -\partial_{w_g, d}^2 C & -\partial_{d, d}^2 C \end{bmatrix}$$

This matrix is, under assumption (1) and (2), negative definite since  $\partial_{w_g, w_g}^2 \mathcal{H}_1 = F''' - \partial_{w_g, w_g}^2 C < 0$  and  $\det(\partial^2 \mathcal{H}_1) = -F''' \cdot \partial_{d, d}^2 C + \left( \partial_{w_g, w_g}^2 C \cdot \partial_{d, d}^2 C - \left( \partial_{w_g, d}^2 C \right)^2 \right) > 0$

### Proof of proposition 2

Point (i) : Existence of a unique steady state

By construction, we know that the steady state satisfies :

$$b(w_g^*, d^*) = 0 \Leftrightarrow \begin{cases} w_g^* = R \\ \underbrace{F'(R) - \partial_w C(R, d^*) - \frac{1}{\rho} \partial_d C_d(R, d^*)}_{\phi(d)} = 0 \end{cases}$$

We therefore only have to check that  $\phi(d) = 0$  admit a unique solution in  $[0, \bar{d}]$ . So let us first observe, by assumption (4) that :

$$\begin{cases} \phi(0) = F'(R) - \partial_w C(R, 0) - \frac{1}{\rho} \partial_d C_d(R, 0) > 0 \\ \phi(\bar{d}) = F'(R) - \partial_w C(R, \bar{d}) - \frac{1}{\rho} \partial_d C_d(R, \bar{d}) < F'(R) - \partial_w C(0, \bar{d}) < 0 \end{cases}$$

Since  $\phi(d)$  is continuous, there exists at least one  $d^* \in ]0, \bar{d}[$  such that  $\phi(d^*) = 0$ . If  $\phi'(d) < 0$ , this one is even unique. This is the case under assumption (2) because :

$$\phi'(d) = -\partial_{w, d}^2 C(R, d) - \frac{1}{\rho} \partial_{d, d}^2 C_d(R, d) < 0$$

*Point (ii)* : The local saddle point dynamic

Since we deal with a two dimensional linear system, we can use the standard results on the trace and the determinant in order to characterize its dynamic. In fact, if the determinant is negative, we deal with a saddle point dynamic. So let us observe that :

$$\begin{aligned}\det(A) &= \begin{bmatrix} \frac{\rho(F'' - \partial_{w_g, w_g}^2 C)}{F'' - \partial_{w_g, w_g}^2 C} \Big|_{(w_g^*, d^*)} & -\frac{\rho \partial_{w_g, d}^2 C + \partial_{d, d}^2 C}{F''' - \partial_{w_g, w_g}^2 C} \Big|_{(w_g^*, d^*)} \\ 1 & 0 \end{bmatrix} \\ &= \frac{\rho \partial_{w_g, d}^2 C + \partial_{d, d}^2 C}{F''' - \partial_{w_g, w_g}^2 C} \Big|_{(w_g^*, d^*)} < 0\end{aligned}$$

## Computations related to remark 1

$$\begin{aligned}& \lim_{\Delta I \rightarrow 0^+} \frac{\mathcal{J}(\Delta I, 0, t_I) - \mathcal{J}(0, 0, 0)}{\Delta I} \\ &= \lim_{\Delta I \rightarrow 0^+} \frac{1}{\Delta I} \left( \int_0^{t_I} (C(R, d_0) - C(R - \Delta I \cdot t, d_0) - \Theta(\Delta I)) \exp^{-\rho t} dt \right. \\ &\quad \left. + \int_{t_I}^\infty (C(R, d_0) - C(R - \Delta I \cdot t_I, d_0)) \exp^{-\rho t} dt \right) \\ &= \int_0^{t_I} (\partial_{w_g} C(R, d_0) \cdot t - \Theta'(0)) \exp^{-\rho t} dt + \int_{t_I}^\infty (\partial_{w_g} C(R, d_0) \cdot t_I) \exp^{-\rho t} dt \\ &= \partial_{w_g} C(R, d_0) \left( \left[ -\frac{t}{\rho} \exp^{-\rho t} \right]_0^{t_I} + \int_0^{t_I} \frac{1}{\rho} \exp^{-\rho t} dt \right) + \partial_{w_g} C(R, d_0) \cdot t_I \left[ -\frac{1}{\rho} \exp^{-\rho t} \right]_{t_I}^\infty - \frac{(1 - \exp^{-\rho t_I})}{\rho} \Theta'(0) \\ &= \frac{1}{\rho} \partial_{w_g} C(R, d_0) \int_0^{t_I} \exp^{-\rho t} dt - \frac{(1 - \exp^{-\rho t_I})}{\rho} \Theta'(0) \\ &= \frac{(1 - \exp^{-\rho t_I})}{\rho} \left( \frac{1}{\rho} \partial_{w_g} C(R, d_0) - \Theta'(0) \right) > 0\end{aligned}$$

## Computations related to remark 2

In a first, we compute the first derivative of the functional 14 with respect to  $t_{w_g}$  and assessed at  $t_{w_g} = 0$ .

$$\begin{aligned}\partial_{t_{w_g}} \mathcal{J}(\Delta I, t_{w_g}, t_I) \Big|_{t_{w_g}=0} &= \left( F(R + \Delta I \cdot t_{w_g}) - C\left(R, d_0 + \Delta I \cdot \frac{t_{w_g}^2}{2}\right) - \Theta(\Delta I) \right) e^{-\rho t_{w_g}} \Big|_{t_{w_g}=0} \\ &\quad - \left( F(R) - C\left(R - \Delta I \cdot t_{w_g}, d_0 + \Delta I \cdot \frac{t_{w_g}^2}{2}\right) - \Theta(\Delta I) \right) e^{-\rho t_{w_g}} \Big|_{t_{w_g}=0} \\ &\quad - \int_{t_{w_g}}^{t_I} \left( \partial_d C\left(R - \Delta I \cdot t, d_0 + \Delta I \cdot \frac{t_{w_g}^2}{2}\right) \cdot \Delta I \cdot t_{w_g} \right) \exp^{-\rho t} dt \Big|_{t_{w_g}=0} \\ &\quad - \int_{t_I}^\infty \left( \partial_d C\left(R - \Delta I \cdot t_I, d_0 + \Delta I \cdot \frac{t_{w_g}^2}{2}\right) \cdot \Delta I \cdot t_{w_g} \right) \exp^{-\rho t} dt \Big|_{t_{w_g}=0} \\ &= (F(R) - C(R, d_0) - \Theta(\Delta I)) - (F(R) - C(R, d_0) - \Theta(\Delta I)) = 0\end{aligned}$$

With the same method, we now compute the second derivative with respect to  $t_{w_g}$  and assessed at  $t_{w_g} = 0$ . After a tedious derivation, we obtain :

$$\begin{aligned}
& \partial_{t_{w_g}, t_{w_g}}^2 \mathcal{J}(\Delta I, t_{w_g}, t_I) \Big|_{t_{w_g}=0} \\
&= F'(R) \cdot \Delta I - \rho [F(R) - C(R, d_0) - \Theta(\Delta I)] - \partial_{w_g} C(R, d_0) \cdot \Delta I + \rho (F(R) - C(R, d_0) - \Theta(\Delta I)) \\
&- \int_0^{t_I} (\partial_d C(R - \Delta I \cdot t, d_0) \cdot \Delta I) \exp^{-\rho t} dt - \int_{t_I}^{\infty} (\partial_d C(R - \Delta I \cdot t_I, d_0) \cdot \Delta I) \exp^{-\rho t} dt \\
&= \Delta I \cdot \left( F'(R) - \partial_{w_g} C(R, d_0) - \int_0^{t_I} \partial_d C(R - \Delta I \cdot t, d_0) \exp^{-\rho t} dt + \frac{1}{\rho} \partial_d C(R - \Delta I \cdot t_I, d_0) \cdot \exp^{-\rho t_I} \right)
\end{aligned}$$

Now remember that  $d_0$  was set at the steady state of the system without harvesting. We therefore know that  $F'(R) - \partial_{w_g} C(R, d_0) = \frac{1}{\rho} \partial_d C(R, d_0)$ . Since  $\partial_d C(R - \Delta I \cdot t, d_0)$  is decreasing in  $t$  because  $\partial_{w_g; d}^2 C > 0$ , we can therefore say that :

$$\begin{aligned}
& \partial_{t_{w_g}, t_{w_g}}^2 \mathcal{J}(\Delta I, t_{w_g}, t_I) \Big|_{t_{w_g}=0} \\
& \geq \underbrace{\Delta I \cdot \left( \frac{1}{\rho} \partial_d C(R, d_0) - \partial_d C(R, d_0) \int_0^{t_I} \exp^{-\rho t} dt - \frac{1}{\rho} \partial_d C(R - \Delta I \cdot t_I, d_0) \cdot \exp^{-\rho t_I} \right)}_{=A}
\end{aligned}$$

and we can observe since  $\partial_{w_g; d}^2 C > 0$ , that the right-hand side term is positive.

$$\begin{aligned}
A &= \Delta I \cdot \left( \frac{1}{\rho} \partial_d C(R, d_0) - \partial_d C(R, d_0) \left[ -\frac{1}{\rho} \exp^{-\rho t} \right]_0^{t_I} - \frac{1}{\rho} \partial_d C(R - \Delta I \cdot t_I, d_0) \cdot \exp^{-\rho t_I} \right) \\
&= \frac{\Delta I}{\rho} \cdot (\partial_d C(R, d_0) + \partial_d C(R - \Delta I \cdot t_I, d_0)) \cdot \exp^{-\rho t_I} > 0
\end{aligned}$$

### Proof of lemma 3

Since we deal with a model with one mixed and four pure state constraints, we normally have to check that (see Grass and al. ([12]) th 3.60) :

$$Q_1 = [\partial_{w_g, I}(w_g), w_g] \text{ and } Q_2 = \begin{bmatrix} \partial_{\omega_g, I}(\partial_d(d) \cdot \dot{d} + \partial_{w_s}(d) \cdot \dot{w}_s) & d & 0 & 0 & 0 \\ \partial_{\omega_g, I}(\partial_d(\bar{d} - d) \cdot \dot{d} + \partial_{w_s}(\bar{d} - d) \cdot \dot{w}_s) & 0 & \bar{d} - d & 0 & 0 \\ \partial_{\omega_g, I}(\partial_d(w_s) \cdot \dot{d} + \partial_{w_s}(w_s) \cdot \dot{w}_s) & 0 & 0 & w_s & 0 \\ \partial_{\omega_g, I}(\partial_d(R - w_s) \cdot \dot{d} + \partial_{w_s}(R - w_s) \cdot \dot{w}_s) & 0 & 0 & 0 & R - w_s \end{bmatrix}$$

are both of full rank. So let us observe that :

$$Q_1 = \begin{bmatrix} 1 & 0 & w_g \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 1 & 0 & d & 0 & 0 & 0 \\ -1 & 0 & 0 & \bar{d} - d & 0 & 0 \\ 0 & 1 & 0 & 0 & w_s & 0 \\ 0 & -1 & 0 & 0 & 0 & R - w_s \end{bmatrix}$$

$Q_1$  is obviously of full rank. Concerning  $Q_2$ , let us remember  $\bar{d}, R > 0$ . This means that we can always choose a non zero vector when we respectively consider the columns 3, 4 and 5, 6. If we add to this choice the 2 first columns we can conclude that  $Q_2$  is of rank 4.

## Proof of lemma 4

It remains to verify that the Hamiltonian  $\mathcal{H}_2(w_g, I, d, w_s, (p_i)_{i=1}^2)$  is strictly concave in  $(w_g, I, d, w_s)$  and that the different constraints are quasi-concave in these variables. This last condition is always satisfied since our constraints are linear. So let us now compute for each  $(p_i)_{i=1}^2$ , the Hessian of this Hamiltonian. We obtain by taking the following order of the variables  $(w_g, w_s, d, I)$

$$\partial^2 \mathcal{H}_2 = \begin{bmatrix} F''' - \partial_{w_g, w_g}^2 C & F''' & -\partial_{w_g, d}^2 C & 0 \\ F''' & F''' - p_2 \delta'' & 0 & 0 \\ -\partial_{w_g, d}^2 C & 0 & -\partial_{d, d}^2 C & 0 \\ 0 & 0 & 0 & -\Theta'' \end{bmatrix}$$

By keeping in mind that  $\Theta'(I) = p_2$ , we observe, under assumption (1), (2) and (3), that :

$$\begin{aligned} D_1 &= F''' - \partial_{w_g, w_g}^2 C < 0 \\ D_2 &= \begin{vmatrix} F''' - \partial_{w_g, w_g}^2 C & F''' \\ F''' & F''' - \Theta' \delta'' \end{vmatrix} = -F''' (\partial_{w_g, w_g}^2 C + \Theta' \delta'') + \Theta' \delta'' \partial_{w_g, w_g}^2 C > 0 \\ D_3 &= \begin{vmatrix} F''' - \partial_{w_g, w_g}^2 C & F''' & -\partial_{w_g, d}^2 C \\ F''' & F''' - \Theta' \delta'' & 0 \\ -\partial_{w_g, d}^2 C & 0 & -\partial_{d, d}^2 C \end{vmatrix} \\ &= F''' \Theta' \delta'' \partial_{d, d}^2 C + (F''' - \Theta' \delta'') (\partial_{w_g, w_g}^2 C \partial_{d, d}^2 C - (\partial_{w_g, d}^2 C)^2) < 0 \end{aligned}$$

and

$$D_4 = |\partial^2 \mathcal{H}_2| = -\Theta'' D_3 > 0$$

It follows that the Hamiltonian is strictly concave for all  $(p_i)_{i=1}^2$ .

## Proof of proposition 3

*Point (i) :* along an optimal path,  $w_g(t) + w_s(t) > 0$

This point follows directly from (ii) of lemma 1. In fact if  $w_s(t) = 0$ , or in other words if there is no harvesting capacity, the control  $w_g(t)$  is, like in section 3, strictly positive.

*Point (ii) :* along an optimal path  $\nexists ]t_0, t_1[ \forall t \in ]t_0, t_1[, w_s(t) = 0$

Assume the contrary. In this case we can immediately say that  $\forall t \in ]t_0, t_1[, (i) q_5(t) = 0$  since the upper bound of  $w_g(t)$  is not reached, (ii)  $I(t) = \delta(0) = 0$  (see assumption (3)) since by the dynamic of the capacity accumulation  $\dot{w}_s(t) = 0$ , (iii)  $p_2(t) = \theta'(0) = 0$  (by assumption (3)) since  $\partial_I \mathcal{L}_2 = 0$ , it follows that  $\dot{p}_2(t) = 0$  and (iv)  $w_g(t) > 0$  (by point (i)) which implies that  $q_1(t) = 0$ .

In order to obtain our contradiction, let us now use these observations and compute :

$$\begin{cases} \partial_{w_g} \mathcal{L}_2 = 0 \\ \rho p_2(t) - \partial_{w_s} \mathcal{L}_2 = \dot{p}_2(t) \end{cases} \Leftrightarrow \begin{cases} F'(w_s(t)) - \partial_{w_g} C(w_g(t), d(t)) + p_1(t) = 0 \\ -F'(w_s(t)) - p_1(t) - q_4(t) = 0 \end{cases}$$

By adding the two previous equations, we obtain  $q_4(t) = -\partial_{w_g} C(w_g(t), d(t)) < 0$  but this contradicts the slackness conditions.

*Point (iii) :*  $\nexists ]t_0, t_1[ \forall t \in ]t_0, t_1[, I(t) \leq 0$  and  $w_g(t) > 0$

Assume the contrary, therefore  $q_1(t) = 0$ . Moreover, by the previous point we know that  $w_s(t) > 0$  almost everywhere in  $]t_0, t_1[$  (a.e. for short) and therefore that  $q_4(t) = 0$  a.e in  $]t_0, t_1[$ . We can even go a step further by saying that  $q_5(t) = 0$  a.e in  $]t_0, t_1[$ . If this is not the case  $\exists ]t_2, t_3[ \subset ]t_0, t_1[, q_5(t) > 0$  and therefore  $w_s(t) = R$ . But, by the dynamics of the harvesting capacity we should have

that  $\dot{w}_s(t) = I(t) - \delta(R) = 0$  which implies that  $I(t) = \delta(R) > 0$ . Finally, by assumption (3) and by  $\partial_I \mathcal{L}_2 = 0$ , we can assert that  $\forall t \in ]t_0, t_1[, p_2(t) = \theta'(I) = 0$  hence that  $\dot{p}_2(t) = 0$ .

These observations lead us to the conclusion that :

$$\begin{aligned} \begin{cases} \partial_{w_g} \mathcal{L}_2 = 0 \\ \rho p_2(t) - \partial_{w_s} \mathcal{L}_2 = \dot{p}_2(t) \end{cases} &\Leftrightarrow \begin{cases} F'(w_s(t) + w_g(t)) - \partial_{w_g} C(w_g(t), d(t)) + p_1(t) = 0 \\ -F'(w_s(t) + w_g(t)) - p_1(t) = 0 \end{cases} \\ &\Rightarrow \partial_{w_g} C(w_g(t), d(t)) = 0 \end{aligned}$$

Since  $\partial_{w_g} C > 0$ , except for  $(w_g(t), d(t)) = (0, 0)$ , we contradict our assumptions.

*Point (iv) :*  $\nexists ]t_0, t_1[ \forall t \in ]t_0, t_1[, w_g(t) = 0$  and  $w_s(t) = R$ .

Assume the contrary. Since  $w_s(t) = R$ , most of the variable related to water harvesting can be computed easily. We obtain  $I(t) = \delta(R)$ ,  $p_2(t) = \Theta'(\delta(R))$  and  $\dot{p}_2(t) = 0$

In order to obtain our contradiction, let us use these observations and compute :

$$\begin{aligned} \begin{cases} \partial_{w_g} \mathcal{L}_2 = 0 \\ \rho p_2(t) - \partial_{w_s} \mathcal{L}_2 = \dot{p}_2(t) \end{cases} &\Leftrightarrow \begin{cases} F'(R) - \partial_{w_g} C(0, d(t)) + p_1(t) + q_1(t) = 0 \\ \Theta'(\delta(R))(\rho + \delta'(R)) - F'(R) - p_1(t) + q_5(t) = 0 \end{cases} \\ \Rightarrow q_1(t) + q_5(t) &= -\Theta'(\delta(R))(\rho + \delta'(R)) + \partial_{w_g} C(0, d(t)) < -\Theta'(\delta(R))(\rho + \delta'(R)) + \partial_{w_g} C(0, \bar{d}) \end{aligned}$$

But by assumption (4), we conclude that  $q_1(t) + q_5(t) < 0$ , a contradiction.

*Point (v) :*  $\nexists ]t_0, t_1[ \forall t \in ]t_0, t_1[, \dot{w}_s(t) = 0$  and either  $d_0 = 0$  or  $d_0 = \bar{d}$  ( and so  $\dot{d}(t) = 0$ )

Assume the contrary. So let  $w_s^0, w_g^0$  stand for the constant values of  $w_s(t)$  and  $w_g(t)$  with  $w_s^0 + w_g^0 = R$ . In this case, we also observe that  $I(t) = \delta(w_s^0)$ ,  $p_2 = \Theta'(\delta(w_s^0))$  so that  $\dot{p}_2(t) = 0$

Now let us remark that :

$$\begin{aligned} \begin{cases} \partial_{w_g} \mathcal{L}_2 = 0 \\ \rho p_2(t) - \partial_{w_s} \mathcal{L}_2 = \dot{p}_2(t) \end{cases} &\Leftrightarrow \begin{cases} F'(R) - \partial_{w_g} C(w_g^0, d_0) + p_1(t) + q_1(t) = 0 \\ \Theta'(\delta(w_s^0))(\rho + \delta'(w_s^0)) - F'(R) - p_1(t) + q_5(t) = 0 \end{cases} \\ \Rightarrow q_1(t) + q_5(t) &= \underbrace{-\Theta'(\delta(w_s^0))(\rho + \delta'(w_s^0)) + \partial_{w_g} C(w_g^0, d_0)}_{=\phi(w_g^0, w_s^0)} \end{aligned}$$

Now remember that  $q_1(t) + q_5(t) \geq 0$  and  $w_s^0 + w_g^0 = R$ , so that  $\phi(R - w_s^0, w_s^0) \geq 0$  with  $w_s^0 \in [0, R]$ . Moreover under assumption (2) and (3) :

$$\frac{d\phi}{dw_s^0}(R - w_s^0, w_s^0) = -\partial_{w_g, w_g}^2 C(R - w_s^0, d_0) - \Theta''(\delta(w_s^0))\delta'(w_s^0)(\rho + \delta'(w_s^0)) - \Theta'(\delta(w_s^0))\delta''(w_s^0) < 0$$

and under either the fact  $\partial_{w_g} C(0, 0) = 0$  for  $d_0 = 0$  or assumption (4) for  $d_0 = \bar{d}$

$$\lim_{w_s^0 \rightarrow R} \phi(R - w_s^0, w_s^0) = -\Theta'(\delta(R))(\rho + \delta'(R)) + \partial_{w_g} C(0, d_0) < 0$$

This means that  $q_1(t) + q_5(t) \geq 0$  is only true for  $w_s^0 < R$ . But this implies by the slackness conditions that  $q_5(t) = 0$ , and since  $w_s^0 + w_g^0 = R$ , then  $w_g^0 > 0$  and therefore  $q_1(t) = 0$ . Moreover by  $\partial_{w_g} \mathcal{L}_2 = 0$  we can even argue that  $p_1(t)$  is a constant hence  $\dot{p}_1(t) = 0$ .

In order to obtain our contradiction, let us wrap all these observations and compute :

$$\rho p_1(t) - \partial_d \mathcal{L}_2 = \dot{p}_1(t) \Leftrightarrow \rho(-F'(R) + \partial_{w_g} C(w_g^0, d_0)) + \partial_d C(w_g^0, d_0) - q_2(t) + q_3(t) = 0$$

Since both constraints on the aquifer level cannot be binding simultaneously, we can say, under assumption (2) and (4) that :

$$\begin{cases} \text{for } d = 0, q_2(t) = \rho(\partial_{w_g} C(w_g^0, 0) - F'(R)) + \partial_d C(w_g^0, 0) < \rho(\partial_{w_g} C(R, 0) - F'(R)) + \partial_d C(R, 0) < 0 \\ \text{for } d = \bar{d}, q_3(t) = \rho(F'(R) - \partial_{w_g} C(w_g^0, \bar{d})) - \partial_d C(w_g^0, \bar{d}) < \rho(F'(R) - \partial_{w_g} C(0, \bar{d})) < 0 \end{cases}$$

But this contradicts the slackness conditions.

## Proof of lemma 5

Let us check that the solutions to

$$\begin{cases} F'(R) - \partial_{w_g} C(w_g^*, d^*) - \frac{1}{\rho} \partial_d C(w_g^*, d^*) = \frac{1}{\rho} (q_3^* - q_2^*) - q_1^* \\ \Theta'(\delta(R - w_g^*)) (\rho + \delta'(R - w_g^*)) - \partial_{w_g} C(w_g^*, d^*) = q_4^* - q_5^* - q_1^* \\ q_1^* \cdot w_g^* = 0, q_2^* \cdot d^* = 0, q_3^* \cdot (\bar{d} - d^*) = 0, q_4^* \cdot (R - w_g^*) = 0 \text{ and } q_5^* \cdot w_g^* = 0 \end{cases}$$

have the property that  $(w_g^*, d^*) \in ]0, R[ \times ]0, \bar{d}[$  which implies that  $(q_i^*)_{i=1}^5 = 0$ . So let us assume the contrary and let us first observe that :

- if  $w_g^* = 0$ , then  $q_4^* = 0$  and the following contradiction is obtained under assumption (4) that :

$$q_5^* + q_1^* = \partial_{w_g} C(0, d^*) - \Theta'(\delta(R)) (\rho + \delta'(R)) < \partial_{w_g} C(0, \bar{d}) - \Theta'(\delta(R)) (\rho + \delta'(R)) < 0$$

- if  $w_g^* = R$ , then  $q_1^* = q_5^* = 0$  and under assumptions (2) and (3), the following contradiction comes out :

$$q_4^* = \Theta'(\delta(0)) (\rho + \delta'(0)) - \partial_{w_g} C(R, d^*) = -\partial_{w_g} C(R, d^*) < 0$$

Up to now, we know that  $w_g^* \in ]0, R[$ , we can therefore set  $q_1^* = q_4^* = q_5^* = 0$ . So let us now observe that :

- if  $d^* = 0$ , then  $q_3^* = 0$  and under assumptions (2) and (4), we have that :

$$\frac{1}{\rho} q_2^* = \partial_{w_g} C(w_g^*, 0) + \frac{1}{\rho} \partial_d C(w_g^*, 0) - F'(R) < \partial_{w_g} C(R, 0) + \frac{1}{\rho} \partial_d C(R, 0) - F'(R) < 0$$

- if  $d^* = \bar{d}$ , then  $q_2^* = 0$  and under the same assumptions as before we can say that :

$$\frac{1}{\rho} q_3^* = F'(R) - \partial_{w_g} C(w_g^*, \bar{d}) - \frac{1}{\rho} \partial_d C(w_g^*, \bar{d}) < F'(R) - \partial_{w_g} C(0, \bar{d}) < 0$$

## Proof of proposition 4

Let  $\Phi : [0, R] \times [0, \bar{d}] \rightarrow \mathbb{R}^2$  be defined by :

$$\Phi(w_g, d) = \begin{pmatrix} \Phi_1(w_g, d) \\ \Phi_2(w_g, d) \end{pmatrix} = \begin{pmatrix} -F'(R) + \frac{1}{\rho} \partial_d C(w_g, d) + \partial_{w_q} C(w_g, d) \\ \Theta(\delta(R - w_g)) (\rho + \delta'(R - w_g)) - \partial_{w_q} C(w_g, d) \end{pmatrix}$$

We have to proof that there exists a unique  $(w_g^*, d^*)$  with the property that  $\Phi(w_g^*, d^*) = 0$  and  $(w_g^*, d^*) \in ]0, R[ \times ]0, \bar{d}[$

### The method

The method relies on a degree theory argument. In fact, we know from Hirsch [14] (see also Mas-Colell [22] p 207-208) that if there exists a map  $G : [0, R] \times [0, \bar{d}] \rightarrow \mathbb{R}^2$  with the properties that

- (i)  $G$  admits a unique regular solution (i.e.  $\deg(G) = 1$ ),
- (ii)  $H(w_g, d, \lambda) = \lambda \cdot G(w_g, d) + (1 - \lambda) \cdot \Phi(w_g, d)$ , with  $\lambda \in [0, 1]$ , is a regular homotopy (i.e.  $\partial H(w_g, d, \lambda)$  is of full rank)
- (iii) the 1-manifold  $H^{-1}(0) \subset ]0, R[ \times ]0, \bar{d}[ \times [0, 1]$ ,

then  $F$  admits at least one solution (i.e.  $\deg(\Phi) = 1$ ). Moreover if

- (iv) the index of each solution (i.e the sign of the determinant of  $\partial \Phi(w_g, d)$  at that point) is constant,

we know that the solution is unique.

Step (i) : Construction of  $G(w_g, d)$

Let us first define  $G(w_g, d) = \begin{pmatrix} d - \delta \\ \omega - w_g \end{pmatrix}$  with  $(\omega, \delta) \in ]0, R[ \times ]0, \bar{d}[$  two parameters which will be specified later. It is immediate that  $G(w_g, d) = 0$  admits a unique solution given by  $(w_g, d) = (\omega, \delta)$  and that  $\det(\partial G(\omega, \delta)) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1$ , so  $\deg(G) = 1$

Step (ii) : Existence of a regular homotopy  $H(w_g, d, \lambda)$

Let us first choose  $(\omega, \delta) \in ]0, R[ \times ]0, \bar{d}[$ . In fact a simple computation shows that

$$\forall \lambda > 0, \quad \text{rank}(\partial_{(\omega, \delta)} H(w_g, d, \lambda; (\omega, \delta))) = \text{rank} \left( \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix} \right) = 2$$

It follows by the generic transversality theorem (see Mas-colell [22] p 45) that for almost all  $(\omega, \delta) \in ]0, R[ \times ]0, \bar{d}[$ , and  $\lambda > 0$ ,  $\partial_{(w_g, d, \lambda)} H(w_g, d, \lambda; \omega, \delta)$  is of full rank. So let us fix one of them. It remains to verify that for  $\lambda = 0$ ,  $\partial H(w_g, d, \lambda)$  is also of full rank. If this is true,  $H$  is a regular homotopy. So, by a simple computation :

$$\begin{aligned} \det(\partial \Phi) &= \begin{vmatrix} \frac{1}{\rho} \partial_{d, w_g}^2 C + \partial_{w_q, w_g}^2 C & \frac{1}{\rho} \partial_{d, d}^2 C + \partial_{w_q, d}^2 C \\ \frac{dA}{dw_g} - \partial_{w_q, w_g}^2 C & -\partial_{w_q, d}^2 C \end{vmatrix} \\ &= \frac{1}{\rho} \left( \partial_{w_q, w_g}^2 C \cdot \partial_{d, d}^2 C - (\partial_{d, w_g}^2 C)^2 \right) - \frac{dA}{dw_g} \cdot \left( \frac{1}{\rho} \partial_{d, d}^2 C + \partial_{w_q, d}^2 C \right) \end{aligned}$$

with  $A(w_g) = \Theta(\delta(R - \omega_g))(\rho + \delta'(R - \omega_g))$ . And our assumptions on the cost function (see assumption (2)) and the production of a water harvesting capacity (see assumption (3)) tell us that  $\det(\partial \Phi) > 0$ , which implies that  $\partial H(w_g, d, 0)$  is of full rank.

Step (iii) : The “interiority” of 1-manifold  $H^{-1}(0)$

Let us assume the contrary. This means that there exists a sequence  $(w_g^n, d^n, \lambda^n) \in H^{-1}(0)$  and  $(w_g^n, d^n, \lambda^n) \rightarrow (w_g^0, d^0, \lambda^0) \in (\{0, R\} \times \{0, \bar{d}\}) \times [0, 1]$ . But let us observe that :

- assumption (4) and the fact that  $\partial_{w_g} \Phi_1(w_g, d) > 0$  bring us to the conclusion that :

$$\begin{cases} \forall w_g \in [0, R], \quad \Phi_1(w_g, 0) = -F'(R) + \frac{1}{\rho} \partial_d C(w_g, 0) + \partial_{w_q} C(w_g, 0) < 0 \\ \forall w_g \in [0, R], \quad \Phi_1(w_g, \bar{d}) = -F'(R) + \frac{1}{\rho} \partial_d C(w_g, \bar{d}) + \partial_{w_q} C(w_g, \bar{d}) > 0 \end{cases}$$

- by the same assumption and the fact that  $\partial_d \Phi_2(w_g, d) < 0$ , we can say that :

$$\forall d \in [0, R], \quad \Phi_2(0, d) = \Theta(\delta(R))(\rho + \delta'(R)) - \partial_{w_q} C(0, d) > 0$$

- finally the property of the harvesting technology (assumption (3)) and the fact that  $\partial_{w_q} C(R, d)$  lead to:

$$\forall d \in [0, R], \quad \Phi_2(R, d) = -\partial_{w_q} C(R, d) < 0$$

It follows, from the first observation, that  $\forall (w_g, \lambda) \in [0, R] \times [0, 1]$

$$\begin{cases} H_1(w_g, 0, \lambda) = -\lambda d_1 + (1 - \lambda) \Phi_1(w_g, 0) < 0 \\ H_1(w_g, \bar{d}, \lambda) = \lambda(\bar{d} - d_1) + (1 - \lambda) \Phi_1(w_g, \bar{d}) > 0 \end{cases}$$

It is therefore impossible that there exists a sequence  $(w_g^n, d^n, \lambda^n) \in H^{-1}(0)$  with  $d^n \rightarrow d^0 \in \{0, \bar{d}\}$ . Let us now move to the second and the third observations. We respectively conclude that  $\forall (d, \lambda) \in [0, \bar{d}] \times [0, 1]$

$$\begin{cases} H_2(0, d, \lambda) = \lambda w_1 + (1 - \lambda) \Phi_2(0, d) > 0 \\ H_2(R, d, \lambda) = \lambda(w_1 - R) + (1 - \lambda) \Phi_2(R, d) < 0 \end{cases}$$

Hence  $\sharp(w_g^n, d^n, \lambda^n) \in H^{-1}(0)$  with  $w_g^n \rightarrow w_g^0 \in \{0, R\}$  and this concludes step (iii).  
Step (iv) : the uniqueness of the solution.

At step (ii) we have observed that  $\forall (w_g, d) \in ]0, R[ \times ]0, \bar{d}[$ ,  $\det(\partial\Phi) > 0$ . It follows the the index of each solution is constant (i.e. equal to one). Uniqueness follows.

## **Proof of proposition 5**

Obvious.